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# Modeling and simulation of combustion fronts in porous media

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**Abstract**

We study a model for forward propagation of a combustion front through a porous medium. The reaction involves oxygen and a solid fuel. We assume that this solid fuel depends on the space variable. We also assume that the amount of gas produced by the reaction is equal to the amount of gas consumed by it. By actual solutions, we prove the existence and uniqueness of solution of the model. We show that temperature is non-decreasing function of time. We use the similarity variable to transform the system of partial differential equations, describing the problem under consideration, into a boundary value problem of coupled ordinary differential equations and an efficient numerical technique is implemented to solve the reduced system. The results are presented graphically and discussed. It is discovered that the heat transfer and species consumption are significantly influenced by the Frank–Kamenetskii number.



*Keywords:* Combustion front; Reaction–diffusion–convection system; Solid fuel; Porous media; Similarity solution

## 1. Introduction

Combustion is the exothermic oxidation of a fuel. In the case of a carbon-based compound, the products are primarily carbon dioxide, water and energy.

Combustion fronts in porous media have been studied by many authors during the last few decades. In particular, for combustion processes in oil recovery, models and results of numerical simulations have been presented. One of the first models of combustion in a petroleum reservoir was formulated by Gottfried [1]. The model consists of a system of six partial differential equations describing the flow of oil, water, and gas through the porous medium, together with a chemical reaction between oxygen and oil. Numerical simulations exhibit all of the main thermal and hydrodynamic features of in-situ combustion known from the laboratory, including propagation of the combustion zone, formation of a steam plateau, and formation of water and oil banks. Crookston and Culham [2] presented a general model for thermal recovery processes, as well as associated numerical procedures. In addition to the aspects of combustion processes modeled by Gottfried, they included such aspects as coke formation and oxidation.

These models are nonlinear reaction–diffusion–convection systems derived from the principle of conservation. In vector form in one space dimension, they have the form

$$H(U)_t + F(U)_x = (B(U)U_x)_x + G(U), \quad 0 < x < l,\ t > 0, \tag{1.1}$$

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**Nomenclature**

| | |
|---|---|
| $A$ | Arrhenius constant |
| $c_{pg}$ | heat capacity of gas at constant pressure |
| $E$ | activation energy |
| $a$ | order of the gaseous reaction rate |
| $R$ | gas constant |
| $s$ | mass/weighted stoichiometric coefficients |
| $T$ | temperature |
| $t$ | time |
| $x$ | position |
| $C$ | concentration/mass fraction |
| $D$ | diffusion coefficient |
| $k$ | the flow resistance |
| $Q$ | heat of reaction |
| $v$ | seepage velocity |
| $p$ | pressure |

*Greek Letters*

| | |
|---|---|
| $\lambda$ | thermal conductivity |
| $\alpha$ | effective thermal diffusivity |
| $\rho$ | density |
| $\omega$ | rate of coke consumption in the chemical reaction |
| $\theta$ | dimensionless temperature |
| $\in$ | dimensionless activation energy $\left\{=\frac{RT_0}{E}\right\}$ |
| $\delta$ | Frank–Kamenetskii number $\left\{=\frac{\alpha Q A\left(C_{ox}^0 p^* p'\right)^a C_f^0 C_{ox}{}'^a C_f{}' e^{-\frac{E}{RT_0}}}{\rho^* \rho' c_{pg} \in T_0 v'^2}\right\}$ |

*Subscripts*

| | |
|---|---|
| $sf$ | solid fuel |
| $ox$ | oxygens |
| $g$ | gas |
| $f$ | fuel |
| 0 | initial |
| $b$ | burner |

*Superscripts*

| | |
|---|---|
| $i$ | injection |

where $l$ is the length of the porous medium and $U$ is the vector of unknown quantities, such as temperature and densities. The first and second terms in (1.1) represent, respectively, accumulation and transport by convection of these quantities; the function $G$ represents source terms due to chemical reactions and heat loss; and the term $(B(U)U_x)_x$ represents diffusion of heat, mass, etc. The combustion process is described by the solution of the system (1.1), with suitable initial and boundary conditions.

In the more recent literature, several authors have studied the oxidation of crude oil with air injected in porous media. These include Ayeni [3] who studied thermal runaway phenomena while investigating the reaction of oxygen and hydrogen. He provided useful theorems on such flows. Davies [4] who tracked an in-situ combustion front using thin flame technique. Schecter and Marchesin [5] who constructed a two-phase model for oxidation, involving air or

oxygen and oil that include heat loss to the rock formation. Souza et al. [6] who studied the Riemann problem with forward combustion due to injection of air into a porous medium containing solid fuel. Olayiwola and Ayeni [7] studied the continuity, momentum and coupled nonlinear energy and species convection–diffusion equations describing the in-situ combustion process in porous media. The numerical solution was provided using large activation energy asymptotics and shooting method.

Some recent work identifies combustion fronts with traveling waves, and proves their existence using geometric methods. Da Mota et al. [8,9] used this approach to study combustion fronts in a two-phase (oil and oxygen) model. Combustion fronts were identified with traveling waves connecting an unburned state ahead of the front to a burned state behind it.

In this paper, we study a model for combustion of oxygen and a solid fuel such as coke in a porous medium. As in [7], we assume the fuel depends on the space variable $x$. We also assume that the amount of gas produced by the reaction is equal to the amount of gas consumed by it. We shift and rescale the temperature so that 0 corresponds to the initial temperature of the porous medium $T_0$, which are also take to be the ignition temperature. Physically relevant solutions must have $0 \leq x \leq l$. However, in order to have a clear picture of solution in this region, we consider all solutions in the upper half plane $x \geq 0$. We prove the existence and uniqueness of solution by actual solutions. We also examine the properties of solution. To simulate the flow, we assume that the incoming mixture is at the burner temperature and also we force three gradients at infinity to be zero.

## 2. Model formulation

We consider a horizontal one-dimensional porous medium with an initially available concentration of a solid fuel such as coke. The space variable is $x$, $0 < x < \infty$, and time is $t$, $t > 0$. The chemical reaction in the medium takes the simple form

$$[solid\ reactant] + [gaseous\ reactant] \longrightarrow [gaseous\ product] + [heat]. \tag{2.1}$$

To formulate balance equations, we assume that solid fuel depends on space variable. The state variables depending on $(x, t)$ are temperature $T$, oxygen mass fraction in the gas phase $C_{ox}$, fuel concentration $C_f$, seepage velocity $v$ and pressure $p$.

The following equations are assumed to hold in the porous medium:

**Balance of energy**

$$\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial x} = \frac{\lambda}{\rho_g c_{pg}}\left(\frac{\partial^2 T}{\partial x^2}\right) + \frac{Q\omega}{\rho_g c_{pg}}. \tag{2.2}$$

**Balance of oxygen mass**

$$\frac{\partial\left(\rho_g C_{ox}\right)}{\partial t} + v\frac{\partial\left(\rho_g C_{ox}\right)}{\partial x} = D_{ox}\left(\frac{\partial^2\left(\rho_g C_{ox}\right)}{\partial x^2}\right) - s_{ox}\omega. \tag{2.3}$$

**Balance of fuel mass**

$$\frac{\partial\left(\rho_f C_f\right)}{\partial t} + v\frac{\partial\left(\rho_f C_f\right)}{\partial x} = D_f\left(\frac{\partial^2\left(\rho_f C_f\right)}{\partial x^2}\right) - \left(1 - s_{sf}\right)\omega. \tag{2.4}$$

**Balance of total gas mass**

$$\frac{\partial \rho_g}{\partial t} + \frac{\partial\left(\rho_g v\right)}{\partial x} = s_g\omega. \tag{2.5}$$

**Darcy's law**

$$v = -k\frac{\partial p}{\partial x}. \tag{2.6}$$

The rate of coke consumption in the chemical reaction, in medium, is assumed to be given by a version of Arrhenius's law:

$$\omega = \begin{cases} A\,(C_{ox}\,p)^a\,C_f e^{-\frac{E}{RT}}, & \text{for } T > 0 \\ 0, & \text{for } T \leq 0, \end{cases} \tag{2.7}$$

where $A$ is the Arrhenius constant, $E$ the activation energy, $a$ the order of the gaseous reaction rate, and $R$ is the gas constant, $\rho_g$ is the gas density, $\rho_f$ is the fuel density, $\lambda$ is thermal conductivity, $c_{pg}$ is heat capacity of gas at constant pressure, $\omega$ is rate of coke consumption in the chemical reaction, $Q$ is heat of reaction, $s_{ox}$ is the mass-weighted stoichiometric coefficients for oxygen, $s_{sf}$ is the mass-weighted stoichiometric coefficients for solid fuel, $s_g$ is the mass-weighted stoichiometric coefficients for gas, $k$ is the flow resistance, which is directly proportional to rock permeability and inversely proportional to gas viscosity.

The quantity $s_g$ may be positive, negative, or zero, depending on whether the amount of gas produced by the reaction is more than, less than, or equal to the amount of gas consumed by it.

Here, we will assume $s_g = 0$, $\rho_g = \rho_f = \rho$.

It is simple to eliminate the balance of total gas mass by means of streamline function,

$$\eta\,(x,t) = \int_0^x \rho\,(s,t)\,ds. \tag{2.8}$$

The coordinate transformation becomes,

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x} = \rho\frac{\partial}{\partial \eta} \tag{2.9}$$

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial t} = -\rho v\frac{\partial}{\partial \eta} + \frac{\partial}{\partial t}. \tag{2.10}$$

We make the additional assumptions that $c_{pg}$, $\rho\lambda$, $\rho^2 D_{ox}$ and $\rho^2 D_f$ are constant. Although these assumptions could be relaxed in the future, they considerably simplify the equations. The equations can be simplified as

$$\frac{\partial T}{\partial t} = \frac{\rho\lambda}{c_{pg}}\frac{\partial^2 T}{\partial \eta^2} + \frac{1}{\rho\,c_{pg}}Q\omega \tag{2.11}$$

$$\frac{\partial C_{ox}}{\partial t} = \rho^2 D_{ox}\frac{\partial^2 C_{ox}}{\partial \eta^2} - \frac{1}{\rho}s_{ox}\omega \tag{2.12}$$

$$\frac{\partial C_f}{\partial t} = \rho^2 D_f\frac{\partial^2 C_f}{\partial \eta^2} - \frac{1}{\rho}\left(1 - s_{sf}\right)\omega \tag{2.13}$$

$$p = constant. \tag{2.14}$$

The initial and boundary conditions were formulated as follows:

Initial condition:

At $t = 0$ and $\forall\,\eta$

$$T = T_0, \qquad C_{ox} = 0, \qquad C_f = 0. \tag{2.15}$$

Boundary conditions:

$$\left.\begin{array}{l} T|_{\eta=0} = T_b, \qquad C_{ox}|_{\eta=0} = C_0, \qquad C_f\big|_{\eta=0} = C_{f0} \\ \lim_{\eta\to\infty}\dfrac{\partial T}{\partial \eta} = \lim_{\eta\to\infty}\dfrac{\partial C_{ox}}{\partial \eta} = \lim_{\eta\to\infty}\dfrac{\partial C_f}{\partial \eta} = 0 \end{array}\right\}, \tag{2.16}$$

where the subscript $b$ means burner.

## 3. Method of solution

### 3.1. Existence and uniqueness of solution

**Theorem 1.** *Let* $\rho^2 D_f = \rho^2 D_{ox} = \frac{\rho\lambda}{c_{pg}} = \lambda_1, s_{ox} = 1 - s_{sf}$. *Then there exists a unique solution of problem* (2.11)–(2.13) *satisfy* (2.15) *and* (2.16).

**Proof.** Let $\rho^2 D_f = \rho^2 D_{ox} = \frac{\rho\lambda}{c_{pg}} = \lambda_1$ and $s_{ox} = 1 - s_{sf}$.

Then, we multiply (2.11) by $s_{ox}$, (2.12) and (2.13) by $\frac{Q}{2c_{pg}}$ and adding, we obtain

$$\frac{\partial \phi}{\partial t} = \lambda_1 \frac{\partial^2 \phi}{\partial \eta^2} \tag{3.1}$$

$$\begin{aligned} &\phi(\eta, 0) = s_{ox} T_0, \qquad \phi(0, t) = s_{ox} T_b + \frac{Q}{2c_{pg}} \left(C_0 + C_{f0}\right), \\ &\frac{\partial}{\partial \eta} \phi(\eta, t) \to 0 \quad \text{as } \eta \to \infty \end{aligned} \tag{3.2}$$

where $\phi = \left(s_{ox} T + \frac{Q}{2c_{pg}} \left(C_{ox} + C_f\right)\right)$.

Using Laplace transform, we obtain the solution of problem (3.1) as

$$\phi(\eta, t) = \left(s_{ox}\left(T_b - T_0\right) + \frac{Q}{2c_{pg}} \left(C_0 + C_{f0}\right)\right) erfc\left(\frac{\eta}{2\sqrt{\lambda_1 t}}\right) + s_{ox} T_o. \tag{3.3}$$

Then, we obtain

$$\begin{aligned} T(\eta, t) \;=\; &\left(\left(T_b - T_0\right) + \frac{Q}{2 s_{ox} c_{pg}} \left(C_0 + C_{f0}\right)\right) erfc\left(\frac{\eta}{2\sqrt{\lambda_1 t}}\right) \\ &+ T_o - \frac{Q}{2 s_{ox} c_{pg}} \left(C_{ox}(\eta, t) + C_f(\eta, t)\right) \end{aligned} \tag{3.4}$$

$$\begin{aligned} C_{ox}(\eta, t) \;=\; &\left(\frac{2 s_{ox} c_{pg}}{Q} \left(T_b - T_0\right) + \left(C_0 + C_{f0}\right)\right) erfc\left(\frac{\eta}{2\sqrt{\lambda_1 t}}\right) \\ &+ \frac{2 s_{ox} c_{pg}}{Q} T_o - \left(\frac{2 s_{ox} c_{pg}}{Q} T(\eta, t) + C_f(\eta, t)\right) \end{aligned} \tag{3.5}$$

$$\begin{aligned} C_f(\eta, t) \;=\; &\left(\frac{2 s_{ox} c_{pg}}{Q} \left(T_b - T_0\right) + \left(C_0 + C_{f0}\right)\right) erfc\left(\frac{\eta}{2\sqrt{\lambda_1 t}}\right) \\ &+ \frac{2 s_{ox} c_{pg}}{Q} T_o - \left(\frac{2 s_{ox} c_{pg}}{Q} T(\eta, t) + C_{ox}(\eta, t)\right). \end{aligned} \tag{3.6}$$

Hence, there exists a unique solution of problem (2.11)–(2.13). This completes the proof.

### 3.2. Non-dimensionalization

We scale the length by using $\eta^* = \frac{\alpha}{v^i}$ and the time by using $t^* = \frac{\eta^*}{v^i}$, where $v^i$ is the injection velocity and $\alpha$ the effective thermal diffusivity. We introduce dimensionless variables for space and time,

$$\eta' = \frac{\eta}{\eta^*}, \qquad t' = \frac{t}{t^*}. \tag{3.7}$$

We also introduce dimensionless variables for temperature, fuel concentration, oxygen mass fraction, pressure and density;

$$\theta = \frac{E}{R T_0^2}\left(T - T_0\right), \qquad p' = \frac{p}{p^*}, \qquad \rho' = \frac{\rho}{\rho^*}, \qquad C_{ox}{}' = \frac{C_{ox}}{C_{ox}^0}, \qquad C_f{}' = \frac{C_f}{C_f^0} \tag{3.8}$$

where $T_0$ is the initial temperature of the porous medium and $C_f^0, C_{ox}^0$ are the initial fuel concentration and initial oxygen mass fraction, respectively; $\eta^*, t^*, p^*, \rho^*$ are reference values for space, time, pressure and density, respectively.

Using (3.5) and (3.6), with these dimensionless variables, and after dropping the prime, the system (2.11)–(2.16) become

$$\frac{\partial\theta}{\partial t} = \rho_1 \frac{\partial^2\theta}{\partial\eta^2} + \delta \left[\gamma\, erfc\left(\frac{\eta}{2\sqrt{\lambda_1 t}}\right)\right]^a \left[\gamma\, erfc\left(\frac{\eta}{2\sqrt{\lambda_1 t}}\right)\right] e^{\frac{\theta}{1+\epsilon\theta}} \tag{3.9}$$

$$\frac{\partial C_{ox}}{\partial t} = \beta_1 \frac{\partial^2 C_{ox}}{\partial\eta^2} - \beta \left[\gamma\, erfc\left(\frac{\eta}{2\sqrt{\lambda_1 t}}\right)\right]^a \left[\gamma\, erfc\left(\frac{\eta}{2\sqrt{\lambda_1 t}}\right)\right] e^{\frac{\theta}{1+\epsilon\theta}} \tag{3.10}$$

$$\frac{\partial C_f}{\partial t} = \sigma_1 \frac{\partial^2 C_f}{\partial\eta^2} - \sigma \left[\gamma\, erfc\left(\frac{\eta}{2\sqrt{\lambda_1 t}}\right)\right]^a \left[\gamma\, erfc\left(\frac{\eta}{2\sqrt{\lambda_1 t}}\right)\right] e^{\frac{\theta}{1+\epsilon\theta}} \tag{3.11}$$

$$\left.\begin{array}{llll} \theta(\eta,0)=0, & \theta(0,t)=\theta_b, & \dfrac{\partial\theta(\eta,t)}{\partial\eta}\to 0 & \text{as } \eta\to\infty \\ C_{ox}(\eta,0)=0, & C_{ox}(0,t)=C_0, & \dfrac{\partial C_{ox}(\eta,t)}{\partial\eta}\to 0 & \text{as } \eta\to\infty \\ C_f(\eta,0)=0, & C_f(0,t)=C_{f0}, & \dfrac{\partial C_f(\eta,t)}{\partial\eta}\to 0 & \text{as } \eta\to\infty \end{array}\right\} \tag{3.12}$$

where

$$\rho_1 = \left(\rho^*\rho'\right)^2, \qquad \beta_1 = \frac{\rho^2}{Le_1}, \qquad \sigma_1 = \frac{\rho^2}{Le_2}, \qquad \epsilon = \frac{RT_0}{E}, \qquad \upsilon = \frac{2s_{ox}c_{pg}\,\epsilon\, T_0}{Q},$$

$$\gamma = \frac{2s_{ox}c_{pg}}{Q}\left(T_b - T_0\right) + \left(C_0 + C_{f0}\right), \qquad \beta = \frac{\alpha s_{ox} A \left(C_{ox}^0 p^* p'\right)^a C_f^0 {C_{ox}}'^a {C_f}' e^{-\frac{E}{RT_0}}}{\rho^*\rho' C_{ox}^0 \upsilon^{i2}},$$

$$\sigma = \frac{\alpha\left(1-s_{sf}\right) A \left(C_{ox}^0 p^* p'\right)^a C_f^0 {C_{ox}}'^a {C_f}' e^{-\frac{E}{RT_0}}}{\rho^*\rho' C_f^0 \upsilon^{i2}}, \qquad \delta = \frac{\alpha Q A \left(C_{ox}^0 p^* p'\right)^a C_f^0 {C_{ox}}'^a {C_f}' e^{-\frac{E}{RT_0}}}{\rho^*\rho' c_{pg}\,\epsilon\, T_0 \upsilon^{i2}}.$$

### 3.3. Properties of solution

**Theorem 2.** *Let* $\epsilon > 0, \rho_1 = \lambda_1 = 1$ *and* $a = \upsilon = C_{ox}^0 = 0$ *in* (3.9). *Then* $\frac{\partial\theta}{\partial t} \geq 0$.

*In the proof, we shall make use of following Lemma of Kolodner and Pederson* [10].

**Lemma** (*Kolodner and Pederson [10]*). *Let* $u(x,t) = 0\left(e^{\alpha|x|^2}\right)$ *be a solution on* $R^n \times [0,t)$ *of the differential inequality* $\frac{\partial u}{\partial t} - \Delta u + K(x,t)\,u \geq 0$ *where* $K$ *is bounded from below. If* $u(x,0) \geq 0$, *then* $u(x,t) \geq 0$ *for all* $(x,t) \in R^n \times [0,t_0)$.

**Proof of Theorem 2.** Let $\epsilon > 0, \rho_1 = \lambda_1 = 1$ *and* $a = \upsilon = C_{ox}^0 = 0$ in (3.9). We obtain

$$\frac{\partial\theta}{\partial t} - \frac{\partial^2\theta}{\partial\eta^2} - \delta\left(\gamma\, erfc\left(\frac{\eta}{2\sqrt{t}}\right)\right) e^{\frac{\theta}{1+\epsilon\theta}} = 0.$$

Differentiating with respect to $t$, we have

$$\frac{\partial}{\partial t}\left(\frac{\partial\theta}{\partial t}\right) - \frac{\partial^2}{\partial\eta^2}\left(\frac{\partial\theta}{\partial t}\right) - \left(\delta\,\gamma\, erfc\left(\frac{\eta}{2\sqrt{t}}\right)\left(\frac{1}{1+\epsilon\,\theta}\right)^2 e^{\frac{\theta}{1+\epsilon\theta}}\right)\frac{\partial\theta}{\partial t} = \frac{\delta\,\gamma}{2\sqrt{\pi}\,t^{\frac{3}{2}}}\eta\, e^{-\frac{\eta^2}{4t}}\, e^{\frac{\theta}{1+\epsilon\theta}}.$$

Let

$$u = \frac{\partial \theta}{\partial t}.$$

Then $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial \eta^2} - \left(\delta\,\gamma\; erfc\left(\frac{\eta}{2\sqrt{t}}\right)\left(\frac{1}{1+\in\theta}\right)^2 e^{\frac{\theta}{1+\in\theta}}\right) u \geq 0$ since $\frac{\delta\,\gamma}{2\sqrt{\pi}t^{\frac{3}{2}}}\eta\, e^{-\frac{\eta^2}{4t}}\, e^{\frac{\theta}{1+\in\theta}} \geq 0.$

This can be written as

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial \eta^2} + K\,(\eta, t)\; u \geq 0,$$

where

$$K\,(\eta, t) = -\left(\delta\,\gamma\; erfc\left(\frac{\eta}{2\sqrt{t}}\right)\left(\frac{1}{1+\in\theta}\right)^2 e^{\frac{\theta}{1+\in\theta}}\right).$$

Clearly, $K$ is bounded from below. Hence by Kolodner and Pederson's lemma $u\,(\eta, t) \geq 0$ i.e., $\frac{\partial \theta}{\partial t} \geq 0$. This completes the proof.

**Theorem 3.** *Let* $\in > 0, a = \rho_1 = \lambda_1 = 1$ *and* $C_f^0 = C_{ox}^0 = 0$ *in* (3.9). *Then* $\theta\,(\eta, t) \geq 0$ *for* $(\eta, t) \in (0, \infty) \times (0, t_0)\,, t_0 > 0.$

**Proof.** Let $\in > 0, a = \rho_1 = \lambda_1 = 1$ and $C_f^0 = C_{ox}^0 = 0$ in (3.9). We obtain $\frac{\partial \theta}{\partial t} - \frac{\partial^2 \theta}{\partial \eta^2} + 2\,\delta\,\upsilon\,\gamma\; erfc\left(\frac{\eta}{2\sqrt{t}}\right) e^{\frac{\theta}{1+\in\theta}}\theta = \delta\left(\left(\gamma\; erfc\left(\frac{\eta}{2\sqrt{t}}\right)\right)^2 + \upsilon^2\theta^2\right) e^{\frac{\theta}{1+\in\theta}}.$

That is $\frac{\partial \theta}{\partial t} - \frac{\partial^2 \theta}{\partial \eta^2} + 2\,\delta\,\upsilon\,\gamma\; erfc\left(\frac{\eta}{2\sqrt{t}}\right) e^{\frac{\theta}{1+\in\theta}}\theta \geq 0$ since $\delta\left(\left(\gamma\; erfc\left(\frac{\eta}{2\sqrt{t}}\right)\right)^2 + \upsilon^2\theta^2\right) e^{\frac{\theta}{1+\in\theta}} \geq 0.$

This can be written as

$$\frac{\partial \theta}{\partial t} - \frac{\partial^2 \theta}{\partial \eta^2} + k\,(\eta, t)\,\theta \geq 0,$$

where

$$k\,(\eta, t) = 2\,\delta\,\upsilon\,\gamma\; erfc\left(\frac{\eta}{2\sqrt{t}}\right) e^{\frac{\theta}{1+\in\theta}}.$$

Hence, by Kolodner and Pederson's lemma $\theta\,(\eta, t) \geq 0$. This completes the proof.

### *3.4. Similarity solution*

Here, we consider Eqs. (3.9)–(3.12) and assume $\rho_1 = \sigma_1 = \beta_1 = \lambda_1$. A similarity solution of Eqs. (3.9)–(3.12) is obtained by defining an independent variable $z$ and dependent variables $\phi$, $f$, $g$ as

$$z = \frac{\eta}{2\sqrt{\lambda_1 t}}, \qquad \theta\,(\eta, t) = \phi\,(z)\,, \qquad C_{ox}\,(\eta, t) = g\,(z)\,, \qquad C_f\,(\eta, t) = f\,(z)\,. \tag{3.13}$$

Substituting Eq. (3.13) into Eqs. (3.9)–(3.12), we obtain

$$\frac{d^2\phi}{dz^2} + 2z\frac{d\phi}{dz} + 4\delta\left[\begin{matrix}\gamma_t\; erfc\left(\frac{\eta}{2\sqrt{\lambda_1 t}}\right)\\ -\upsilon_t\phi - C_{ft}^0 f\end{matrix}\right]^a \left[\begin{matrix}\gamma_t\; erfc\left(\frac{\eta}{2\sqrt{\lambda_1 t}}\right)\\ -\upsilon_t\phi - C_{oxt}^0 g\end{matrix}\right] e^{\frac{\phi}{1+\in\phi}} = 0 \tag{3.14}$$

$$\frac{d^2 g}{dz^2} + 2z\frac{dg}{dz} - 4\beta\left[\begin{matrix}\gamma_t\; erfc\left(\frac{\eta}{2\sqrt{\lambda_1 t}}\right)\\ -\upsilon_t\phi - C_{ft}^0 f\end{matrix}\right]^a \left[\begin{matrix}\gamma_t\; erfc\left(\frac{\eta}{2\sqrt{\lambda_1 t}}\right)\\ -\upsilon_t\phi - C_{oxt}^0 g\end{matrix}\right] e^{\frac{\phi}{1+\in\phi}} = 0 \tag{3.15}$$

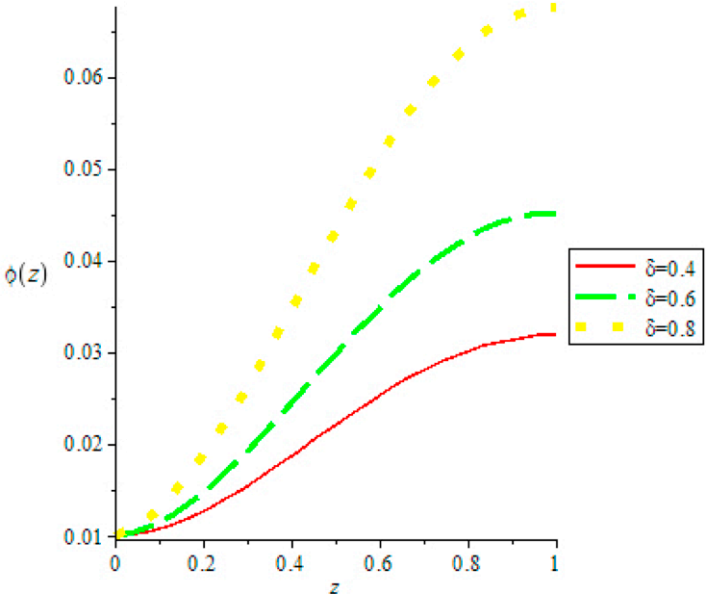


Fig. 1. Plot of $\phi(z)$ against $z$ for different values of $\delta$ and $\gamma_t = 0.22$, $\beta = 0.3$, $\sigma = 0.3$, $\upsilon_t = 0.2$, $\varepsilon = 0.01$, $C^0_{ft} = 1$, $C^0_{0xt} = 1$, $\alpha = 1$.

$$\frac{d^2 f}{dz^2} + 2z\frac{df}{dz} - 4\sigma \left[\begin{matrix}\gamma_t\, erfc\left(\dfrac{\eta}{2\sqrt{\lambda_1 t}}\right)\\ -\upsilon_t\phi - C^0_{ft} f\end{matrix}\right]^a \left[\begin{matrix}\gamma_t\, erfc\left(\dfrac{\eta}{2\sqrt{\lambda_1 t}}\right)\\ -\upsilon_t\phi - C^0_{oxt} g\end{matrix}\right] e^{\frac{\phi}{1+\in\phi}} = 0 \tag{3.16}$$

$$\left.\begin{matrix}\phi(0) = \theta_b, & g(0) = C_0, & f(0) = C_{f0}\\ \phi'(\infty) = 0, & g'(\infty) = 0, & f'(\infty) = 0\end{matrix}\right\} \tag{3.17}$$

where

$$\gamma_t = \gamma, t, \qquad \upsilon_t = \upsilon\, t, \qquad C^0_{ft} = C^0_f\, t, \qquad C^0_{oxt} = C^0_{ox}\, t.$$

It is noteworthy that the parameters $\gamma_t$, $\upsilon_t$, $C^0_{ft}$ and $C^0_{oxt}$ in Eqs. (3.14)–(3.16) are functions of $t$. However, in order to have a similarity solution all the parameters $\gamma_t$, $\upsilon_t$, $C^0_{ft}$, $C^0_{oxt}$ must be constant and we therefore assume

$$\gamma = pt^{-\frac{1}{a+1}}, \qquad \upsilon = qt^{-\frac{1}{a+1}}, \qquad C^0_f = rt^{-\frac{1}{a+1}}, \qquad C^0_{ox} = st^{-\frac{1}{a+1}} \tag{3.18}$$

where $p, q, r, s$ are constants.

### *3.5. Numerical procedure*

The system of non-linear ordinary differential equations (3.14)–(3.16) with boundary conditions (3.17) has been solved numerically using the Runge–Kutta integration scheme with a modified version of the Newton–Raphson shooting method with $\delta, \beta, \sigma, \gamma_t, \upsilon_t, C^0_{ft}$ and $C^0_{oxt}$ as prescribed parameters. The computations were done using computer symbolic algebraic package MAPLE.

## 4. Results and discussion

The existence and uniqueness of solution of the Problem is proved by actual solutions. Also, under certain condition, we have shown that $\theta(x, t)$ is non-decreasing function of time. Numerical solutions of Eqs. (3.14)–(3.17) are computed for the values of $\gamma_t = 0.22$, $\beta = 0.3$, $\sigma = 0.3$, $\upsilon_t = 0.2$, $\in = 0.01$, $C^o_{ft} = C^o_{oxt} = a = 1$. The species concentration and temperature values are depicted graphically in Figs. 1–3.

The temperature distribution behavior is shown in Fig. 1. Fig. 1 depicts the graph of $\phi(z)$ against $z$ for different values of $\delta$. It is observed that the temperature increases as Frank–Kamenetskii number increases. The oxygen mass

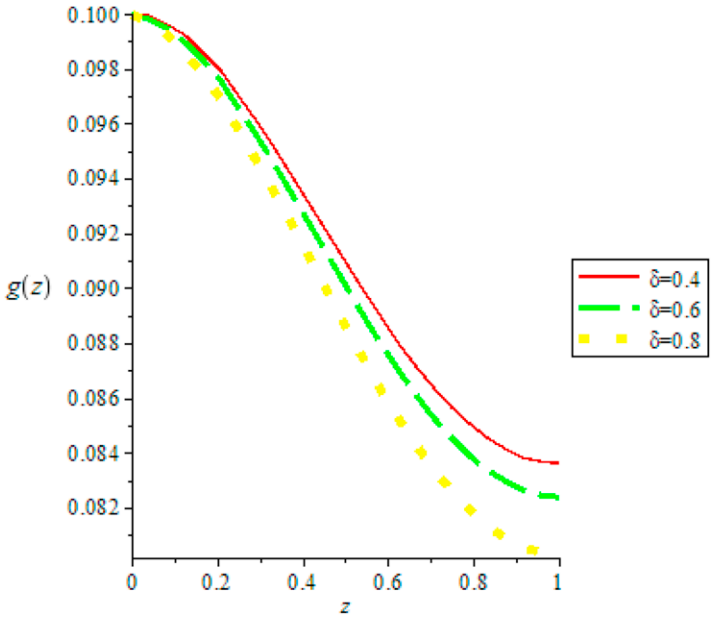


Fig. 2. Plot of $g(z)$ against $z$ for different values of $\delta$ and $\gamma_t = 0.22, \beta = 0.3, \sigma = 0.3, \upsilon_t = 0.2, \varepsilon = 0.01, C^0_{ft} = 1, C^0_{0xt} = 1, \alpha = 1$.

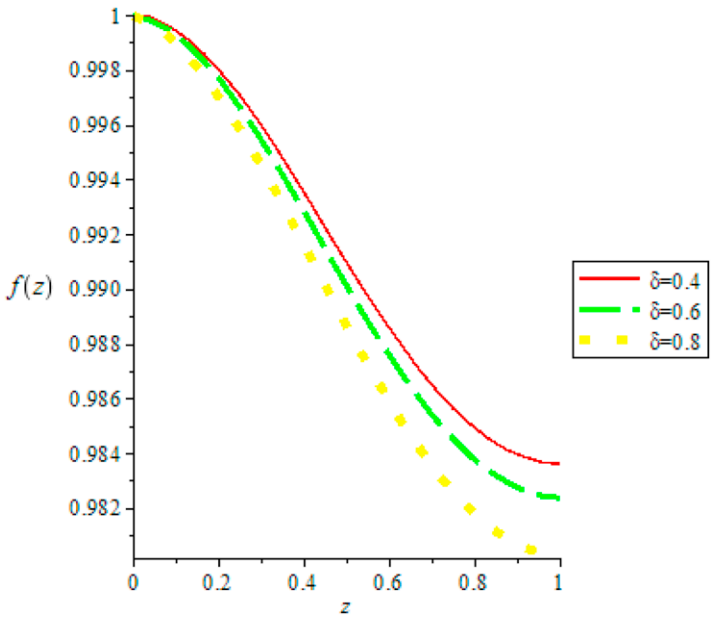


Fig. 3. Plot of $f(z)$ against $z$ for different values of $\delta$ and $\gamma_t = 0.22, \beta = 0.3, \sigma = 0.3, \upsilon_t = 0.2, \varepsilon = 0.01, C^0_{ft} = 1, C^0_{0xt} = 1, \alpha = 1$.

fraction distribution behavior is shown in Fig. 2. Fig. 2 depicts the graph of $g\ (z)$ against $z$ for different values of $\delta$. It is observed that the oxygen mass fraction decreases as Frank–Kamenetskii number increases. The fuel mass fraction distribution behavior is shown in Fig. 3. Fig. 3 depicts the graph of $f\ (z)$ against $z$ for different values of $\delta$. It is observed that the fuel mass fraction decreases as Frank–Kamenetskii number increases.

It is worth pointing out that the effect of $\delta$ as shown in Figs. 1–3 physically means that the temperature is increased and species is consumed. These occur as a result of increase in heat of reaction. When the heat of reaction is high, the rate of conversion of solid fuel into light oils, water and gas is high and consequently, the recovery rate is boosted. This is of great economic importance.

## 5. Conclusion

To study a model for forward propagation of a combustion front through a porous medium, we used similarity variable to transform the system of partial differential equations into a boundary value problem of coupled ordinary differential equations and numerical solution via Runge–Kutta integration scheme with a modified version of the Newton–Raphson shooting method is obtained. The governing parameter for the problem under study is the Frank–Kamenetskii number. The temperature and species mass fraction profiles are significantly influenced by the parameter involved. The results may be used as a preliminary predictive tool to study mathematically the forward propagation of a combustion front through a porous medium. The work may be extended to more complex cases such as when the amount of gas produced by the reaction is more than or less than the amount of gas consumed by it and therefore, recommended for further research.

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